High–energy scattering and Euclidean–Minkowskian duality

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Abstract
We shall discuss how some relevant analyticity and crossing-symmetry properties of the "eikonal scattering amplitudes" of two Wilson loops in QCD, when going from Euclidean to Minkowskian theory, can be related to the still unsolved problem of the asymptotic s–dependence of the hadron–hadron total cross–sections. In particular, we critically discuss the question if (and how) a *pomeron*–like behaviour can be derived from this Euclidean–Minkowskian duality.

1 Loop–loop and meson–meson scattering amplitudes

It was shown in Refs. [1, 2] (for a review see Refs. [3–5] and references therein) that the high–energy meson–meson elastic scattering amplitude can be approximately reconstructed in two steps: i) one first evaluates, in the functional–integral approach, the high–energy elastic scattering amplitude of two $q\bar{q}$ pairs (usually called *dipoles*), of given transverse sizes $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ and given longitudinal–momentum fractions f_1 and f_2 of the two quarks in the two dipoles respectively; ii) one then averages this amplitude over all possible values of $\vec{R}_{1\perp}, f_1$ and $\vec{R}_{2\perp}, f_2$ with two proper squared wave functions $|\psi_1(\vec{R}_{1\perp}, f_1)|^2$ and $|\psi_2(\vec{R}_{2\perp}, f_2)|^2$, describing the two interacting mesons.

The high–energy elastic scattering amplitude of two dipoles (defined in Eq. (8) below) is governed by the following (properly normalized) connected correlation function of two Wilson loops forming an hyperbolic angle χ in the longitudinal plane (see Eq. (4) below) and separated by a distance $\vec{z}_\perp = (z^2, z^3)$ in the transverse plane (impact parameter):

$$\mathcal{C}_M(\chi, \vec{z}_\perp; 1, 2) \equiv \lim_{T\to\infty} \left[\frac{\langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle}{\langle \mathcal{W}_1^{(T)} \rangle \langle \mathcal{W}_2^{(T)} \rangle} - 1 \right], \tag{1}$$

where the arguments "1" and "2" in the function $\mathcal{C}_M$ stand for "$\vec{R}_{1\perp}, f_1$" and "$\vec{R}_{2\perp}, f_2$" respectively and the expectation values $\langle \ldots \rangle$ are averages in the sense of the QCD functional integrals. The two (infrared regularized) Wilson loops $\mathcal{W}_1^{(T)}$ and $\mathcal{W}_2^{(T)}$ are defined as:

$$\mathcal{W}_{1,2}^{(T)} \equiv \frac{1}{N_c} \mathrm{Tr} \left\{ \mathcal{P} \exp \left[-ig \oint_{\mathcal{C}_{1,2}} A_\mu(x) dx^\mu \right] \right\}, \tag{2}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two rectangular paths which follow the classical straight lines for quark $[X_q(\tau)$, forward in proper time $\tau]$ and antiquark $[X_{\bar{q}}(\tau)$, backward in $\tau]$ trajectories, i.e.,

$$\begin{aligned} \mathcal{C}_1 \quad &: \quad X_{1q}^\mu(\tau) = z^\mu + \frac{p_1^\mu}{m}\tau + (1 - f_1) R_1^\mu, \quad X_{1\bar{q}}^\mu(\tau) = z^\mu + \frac{p_1^\mu}{m}\tau - f_1 R_1^\mu, \\ \mathcal{C}_2 \quad &: \quad X_{2q}^\mu(\tau) = \frac{p_2^\mu}{m}\tau + (1 - f_2) R_2^\mu, \quad X_{2\bar{q}}^\mu(\tau) = \frac{p_2^\mu}{m}\tau - f_2 R_2^\mu, \end{aligned} \tag{3}$$

and are closed by straight–line paths at proper times $\tau = \pm T$, where T plays the role of an infrared cutoff, which can and must be removed in the end ($T \to \infty$). Here p_1 and p_2 are the four–momenta of the two dipoles, taken for simplicity with the same mass m, moving (in the center–of–mass system) with speed V and $-V$ along, for example, the x^1–direction:

$$p_1 = m\left(\cosh\frac{\chi}{2}, \sinh\frac{\chi}{2}, \vec{0}_\perp\right), \quad p_2 = m\left(\cosh\frac{\chi}{2}, -\sinh\frac{\chi}{2}, \vec{0}_\perp\right), \tag{4}$$

$\chi = 2\,\mathrm{arctanh}V$ being the hyperbolic angle between the two trajectories $1q$ and $2q$, i.e., $p_1 \cdot p_2 = m^2 \cosh\chi$. Therefore, in terms of the usual Mandelstam variable s:

$$s \equiv (p_1 + p_2)^2 = 2m^2(\cosh\chi + 1)\ , \quad \text{i.e.}:\ \chi \underset{s\to\infty}{\sim} \log\left(\frac{s}{m^2}\right). \tag{5}$$

It is convenient to consider also the correlation function $\mathcal{C}_E(\theta, \vec{z}_\perp; 1, 2)$ in the Euclidean theory of two Euclidean Wilson loops running along two rectangular paths $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$, defined analogously to (3), with the same $\vec{R}_{1\perp}$, $\vec{R}_{2\perp}$, $\vec{z}_\perp$ and with the Minkowskian four–momenta p_1, p_2 replaced by the following Euclidean four–vectors:

$$p_{1E} = m\left(\sin\frac{\theta}{2}, \vec{0}_\perp, \cos\frac{\theta}{2}\right), \quad p_{2E} = m\left(-\sin\frac{\theta}{2}, \vec{0}_\perp, \cos\frac{\theta}{2}\right), \tag{6}$$

θ being the angle formed by the two trajectories $1q$ and $2q$ in Euclidean four–space, i.e., $p_{1E} \cdot p_{2E} = m^2\cos\theta$. It has been proved in Ref. [6] that the Minkowskian quantity $\mathcal{C}_M$ with $\chi \in \mathbb{R}^+$ can be reconstructed from the corresponding Euclidean quantity $\mathcal{C}_E$, with $\theta \in (0, \pi)$, by an analytic continuation in the angular variables $\theta \to -i\chi$, exactly as in the case of Wilson lines [7–9]. This result is derived under certain hypotheses of analyticity in the angular variables [10]. In particular, one makes the assumption that the function $\mathcal{C}_E$, as a function of the *complex* variable θ, can be *analytically extended* from the real segment $(0 < \mathrm{Re}\theta < \pi, \mathrm{Im}\theta = 0)$ to a domain $\mathcal{D}_E$, which also includes the negative imaginary axis $(\mathrm{Re}\theta = 0+, \mathrm{Im}\theta < 0)$; and, therefore, the function $\mathcal{C}_M$, as a function of the *complex* variable χ, can be *analytically extended* from the positive real axis $(\mathrm{Re}\chi > 0, \mathrm{Im}\chi = 0+)$ to a domain $\mathcal{D}_M = \{\chi \in \mathbb{C} \mid -i\chi \in \mathcal{D}_E\}$, which also includes the imaginary segment $(\mathrm{Re}\chi = 0, 0 < \mathrm{Im}\chi < \pi)$. The validity of this assumption is confirmed by explicit calculations in perturbation theory [6, 7, 11]. Denoting with $\overline{\mathcal{C}}_M$ and $\overline{\mathcal{C}}_E$ such analytic extensions, we then have the following *analytic–continuation relations* [6, 10]:

$$\begin{aligned} \overline{\mathcal{C}}_E(\theta, \vec{z}_\perp; 1, 2) &= \overline{\mathcal{C}}_M(i\theta, \vec{z}_\perp; 1, 2), \qquad \forall\theta \in \mathcal{D}_E; \\ \overline{\mathcal{C}}_M(\chi, \vec{z}_\perp; 1, 2) &= \overline{\mathcal{C}}_E(-i\chi, \vec{z}_\perp; 1, 2), \qquad \forall\chi \in \mathcal{D}_M. \end{aligned} \tag{7}$$

The validity of the relation (7) for the loop–loop correlators in QCD has been also recently verified in Ref. [11] by an explicit calculation up to the order $\mathcal{O}(g^6)$ in perturbation theory. However we want to stress that the analytic continuation (7) is expected to be an *exact* result, i.e., not restricted to some order in perturbation theory or to some other approximation, and is valid both for the Abelian and the non–Abelian case.

The relation (7) allows the derivation of the *loop–loop scattering amplitude*, which is defined as

$$\mathcal{M}_{(ll)}(s, t; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2) = -i\ 2s\ \widetilde{\mathcal{C}}_M\left(\chi \underset{s\to\infty}{\sim} \log\left(\frac{s}{m^2}\right), t; 1, 2\right), \tag{8}$$

$\widetilde{\mathcal{C}}_M$ being the two–dimensional Fourier transform of $\mathcal{C}_M$, with respect to the impact parameter $\vec{z}_\perp$, at transferred momentum $\vec{q}_\perp$ (with $t = -|\vec{q}_\perp|^2$), i.e.,

$$\widetilde{\mathcal{C}}_M(\chi, t; 1, 2) \equiv \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \mathcal{C}_M(\chi, \vec{z}_\perp; 1, 2), \tag{9}$$

from the analytic continuation $\theta \to -i\chi$ of the corresponding Euclidean quantity:

$$\widetilde{\mathcal{C}}_E(\theta, t; 1, 2) \equiv \int d^2\vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \mathcal{C}_E(\theta, \vec{z}_\perp; 1, 2), \tag{10}$$

which can be evaluated non-perturbatively by well–known and well–established techniques available in the Euclidean theory. This approach has been extensively used in the literature [12–16] in order to tackle, from a theoretical point of view, the still unsolved problem of the asymptotic s–dependence of hadron–hadron elastic scattering amplitudes and total cross sections. As we have already said in the beginning, the *hadron–hadron elastic scattering amplitude* $\mathcal{M}_{(hh)}$ can be obtained by averaging the loop–loop scattering amplitude (8) over all possible dipole transverse separations $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ and longitudinal–momentum fractions f_1 and f_2 with two proper squared hadron wave functions [1–5]:

$$\begin{aligned} \mathcal{M}_{(hh)}(s,t) &= \int d^2\vec{R}_{1\perp} \int_0^1 df_1\, |\psi_1(\vec{R}_{1\perp}, f_1)|^2 \int d^2\vec{R}_{2\perp} \int_0^1 df_2\, |\psi_2(\vec{R}_{2\perp}, f_2)|^2 \\ &\times \mathcal{M}_{(ll)}(s, t; \vec{R}_{1\perp}, f_1, \vec{R}_{2\perp}, f_2). \end{aligned} \tag{11}$$

Denoting with $\mathcal{C}_M^{(hh)}$ and $\mathcal{C}_E^{(hh)}$ the quantities obtained by averaging the corresponding loop–loop correlation functions $\mathcal{C}_M$ and $\mathcal{C}_E$ over all possible dipole transverse separations $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ and longitudinal–momentum fractions f_1 and f_2, in the same sense as in Eq. (11), we can write:

$$\mathcal{M}_{(hh)}(s,t) = -i\, 2s\, \widetilde{\mathcal{C}}_M^{(hh)} \left(\chi \underset{s\to\infty}{\sim} \log\left(\frac{s}{m^2}\right), t \right). \tag{12}$$

Clearly, by virtue of the relation (7), we also have that:

$$\overline{\widetilde{\mathcal{C}}_M^{(hh)}}(\chi, t) = \overline{\widetilde{\mathcal{C}}_E^{(hh)}}(-i\chi, t), \qquad \forall \chi \in \mathcal{D}_M. \tag{13}$$

By virtue of the optical theorem, the hadron–hadron total cross section can be derived from the imaginary part of the forward hadron–hadron elastic scattering amplitude. Experimental observations at the present time seem to be well described by a *pomeron*–like high–energy behaviour (see, for example, Ref. [4] and references therein):

$$\sigma_{\rm tot}^{(hh)}(s) \underset{s\to\infty}{\sim} \frac{1}{s} {\rm Im}\mathcal{M}_{(hh)}(s, t=0) \sim \sigma_0^{(hh)} \left(\frac{s}{s_0}\right)^{\epsilon_P}, \quad \text{with } \epsilon_P \simeq 0.08. \tag{14}$$

A behaviour like the one of Eq. (14) seems to emerge directly (apart from possible undetermined $\log s$ prefactors) when applying the Euclidean–to–Minkowskian analytic–continuation approach to the study of the line–line/loop–loop scattering amplitudes in strongly coupled (confining) gauge theories using the AdS/CFT correspondence [15, 16].

Moreover, it has been found in Ref. [11] that the dipole–dipole cross section, evaluated from the loop–loop correlator up to the order $\mathcal{O}(g^6)$, reproduces the first iteration of the BFKL *kernel* in the leading–log approximation, the so–called BFKL–*pomeron* behaviour, i.e., $\sim s^{\frac{12\alpha_s}{\pi}\log 2}$, with $\alpha_s = g^2/4\pi$ [17].

2 How a pomeron–like behaviour can be derived

The way in which a *pomeron*–like behaviour can emerge, using the Euclidean–to–Minkowskian analytic continuation, was first shown in Ref. [7] in the case of the line–line (i.e., parton–parton) scattering amplitudes. Here we shall readapt that analysis to the case of the loop–loop scattering amplitudes, with more technical developments, new interesting insights and critical considerations [18]. We start by writing the Euclidean hadronic correlation function in a partial–wave expansion:

$$\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t) = \sum_{l=0}^{\infty}(2l+1)A_l(t)P_l(\cos\theta). \tag{15}$$

As shown in Ref. [10], the loop–antiloop correlator at angle θ in the Euclidean theory (or at hyperbolic angle χ in the Minkowskian theory) can be derived from the corresponding loop–loop correlator by the substitution $\theta \to \pi - \theta$ (or $\chi \to i\pi - \chi$ in the Minkowskian theory). Because of these *crossing–symmetry relations*, it is natural to decompose also our hadronic correlation function $\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t)$ as a sum of a *crossing–symmetric* function $\widetilde{\mathcal{C}}_E^{+}(\theta,t)$ and of a *crossing–antisymmetric* function $\widetilde{\mathcal{C}}_E^{-}(\theta,t)$:

$$\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t) = \widetilde{\mathcal{C}}_E^{+}(\theta,t) + \widetilde{\mathcal{C}}_E^{-}(\theta,t), \quad \widetilde{\mathcal{C}}_E^{\pm}(\theta,t) \equiv \frac{\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t) \pm \widetilde{\mathcal{C}}_E^{(hh)}(\pi-\theta,t)}{2}. \tag{16}$$

Using Eq. (15), we can find the partial–wave expansions of these two functions as follows:

$$\widetilde{\mathcal{C}}_E^{\pm}(\theta,t) = \frac{1}{2}\sum_{l=0}^{\infty}(2l+1)A_l(t)[P_l(\cos\theta) \pm P_l(-\cos\theta)]. \tag{17}$$

Because of the relation $P_l(-\cos\theta) = (-1)^l P_l(\cos\theta)$, valid for non–negative integer values of l, we immediately see that $\widetilde{\mathcal{C}}_E^{+}(\theta,t)$ gets contributions only from even l, while $\widetilde{\mathcal{C}}_E^{-}(\theta,t)$ gets contributions only from odd l. For this reason the functions $\widetilde{\mathcal{C}}_E^{\pm}(\theta,t)$ can also be called *even–signatured* and *odd–signatured* correlation functions respectively and we can replace $A_l(t)$ in Eq. (17) respectively with $A_l^{\pm}(t) \equiv \frac{1}{2}[1 \pm (-1)^l]A_l(t)$. However, if we write the hadronic correlation function $\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t)$ in terms of the loop–loop correlation function, averaged over all possible dipole transverse separations $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ and longitudinal–momentum fractions f_1 and f_2 with two proper squared hadron wave functions $|\psi_1(\vec{R}_{1\perp},f_1)|^2$ and $|\psi_2(\vec{R}_{2\perp},f_2)|^2$, and we make use: i) of the so–called *crossing–symmetry relations* for loop–loop correlators [10]:

$$\begin{aligned}&\mathcal{C}_E(\pi-\theta,\vec{z}_\perp;\vec{R}_{1\perp},f_1,\vec{R}_{2\perp},f_2)\\ &\quad= \mathcal{C}_E(\theta,\vec{z}_\perp;\vec{R}_{1\perp},f_1,-\vec{R}_{2\perp},1-f_2) = \mathcal{C}_E(\theta,\vec{z}_\perp;-\vec{R}_{1\perp},1-f_1,\vec{R}_{2\perp},f_2), \quad \forall\theta\in\mathbb{R};\end{aligned} \tag{18}$$

and ii) of the rotational– and C–invariance of the squared hadron wave functions, that is:

$$|\psi_i(\vec{R}_{i\perp},f_i)|^2 = |\psi_i(-\vec{R}_{i\perp},f_i)|^2 = |\psi_i(\vec{R}_{i\perp},1-f_i)|^2 = |\psi_i(-\vec{R}_{i\perp},1-f_i)|^2 \tag{19}$$

(see Refs. [3, 5] and also [4], chapter 8.6, and references therein), then we immediately conclude that the hadronic correlation function $\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t)$ is automatically crossing symmetric and

so it coincides with the even–signatured function $\widetilde{\mathcal{C}}_E^+(\theta,t)$, the odd–signatured function $\widetilde{\mathcal{C}}_E^-(\theta,t)$ being identically equal to zero. Upon analytic continuation from the Euclidean to the Minkowskian theory (see again Ref. [10]), this means that the Minkowskian hadronic correlation function $\widetilde{\mathcal{C}}_M^{(hh)}(\chi,t)$, and therefore also the scattering amplitude $\mathcal{M}_{(hh)}$ written in Eq. (12), turns out to be automatically crossing symmetric, i.e., invariant under the exchange $\chi \to i\pi - \chi$: $\widetilde{\mathcal{C}}_M^{(hh)}(\chi,t) = \widetilde{\mathcal{C}}_M^+(\chi,t)$, $\widetilde{\mathcal{C}}_M^-(\chi,t) = 0$. In other words, our formalism naturally leads to a high–energy meson–meson scattering amplitude which, being crossing symmetric, automatically satisfies the Pomeranchuk theorem. An *odderon* (i.e., $C=-1$) exchange seems to be excluded for high–energy meson–meson scattering, while a *pomeron* (i.e., $C=+1$) exchange is possible [19].

Let us therefore proceed by considering our *crossing–symmetric* Euclidean correlation function:

$$\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t) = \widetilde{\mathcal{C}}_E^+(\theta,t) = \frac{1}{2}\sum_{l=0}^{\infty}(2l+1)A_l^+(t)[P_l(\cos\theta)+P_l(-\cos\theta)]. \tag{20}$$

We can now use Cauchy's theorem to rewrite this partial–wave expansion as an integral over l, the so–called *Sommerfeld–Watson transform*:

$$\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t) = \widetilde{\mathcal{C}}_E^+(\theta,t) = -\frac{1}{4i}\int_C \frac{(2l+1)A_l^+(t)[P_l(-\cos\theta)+P_l(\cos\theta)]}{\sin(\pi l)}dl, \tag{21}$$

where "C" is a contour in the complex l–plane, running clockwise around the real positive l–axis and enclosing all non–negative integers, while excluding all the singularities of A_l^+. Here (as in the original derivation: see, e.g., Ref. [4] and references therein) we make the fundamental *assumption* that the singularities of $A_l^+(t)$ in the complex l–plane (at a given t) are only *simple poles*. (However, we want to remark that our *partial–wave amplitudes* $A_l^+(t)$ are *not* the same partial–wave amplitudes considered in the original derivation.) Then we can use again Cauchy's theorem to reshape the contour C into the straight line $\mathrm{Re}(l) = -\frac{1}{2}$ and rewrite the integral (21) as follows:

$$\begin{aligned}
&\widetilde{\mathcal{C}}_E^{(hh)}(\theta,t) = \widetilde{\mathcal{C}}_E^+(\theta,t) = \\
&\quad -\frac{\pi}{2}\sum_{\mathrm{Re}(\sigma_n^+)>-\frac{1}{2}} \frac{(2\sigma_n^+(t)+1)r_n^+(t)[P_{\sigma_n^+(t)}(-\cos\theta)+P_{\sigma_n^+(t)}(\cos\theta)]}{\sin(\pi\sigma_n^+(t))} \\
&\quad -\frac{1}{4i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(2l+1)A_l^+(t)[P_l(-\cos\theta)+P_l(\cos\theta)]}{\sin(\pi l)}dl,
\end{aligned} \tag{22}$$

where $\sigma_n^+(t)$ is a pole of $A_l^+(t)$ in the complex l–plane and $r_n^+(t)$ is the corresponding residue. We have also assumed that the large–l behaviour of A_l^+ is such that the integrand function in Eq. (21) vanishes enough rapidly (faster than $1/l$) as $|l| \to \infty$ in the right half–plane, so that the contribution from the infinite contour is zero.

Eq. (22) immediately leads to the asymptotic behaviour of the scattering amplitude in the limit $s \to \infty$, with a fixed t ($|t| \ll s$). In fact, making use of the analytic extension (13) when

continuing the angular variable, $\theta \to -i\chi$, we derive that for every $\chi \in \mathbb{R}^+$:

$$
\begin{aligned}
&\widetilde{\mathcal{C}}_M^{(hh)}(\chi, t) = \overline{\widetilde{\mathcal{C}}_E^{(hh)}}(-i\chi, t) = \\
&-\frac{\pi}{2} \sum_{\mathrm{Re}(\sigma_n^+) > -\frac{1}{2}} \frac{(2\sigma_n^+(t)+1) r_n^+(t) [P_{\sigma_n^+(t)}(-\cosh\chi) + P_{\sigma_n^+(t)}(\cosh\chi)]}{\sin(\pi\sigma_n^+(t))} \\
&-\frac{1}{4i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{(2l+1) A_l^+(t) [P_l(-\cosh\chi) + P_l(\cosh\chi)]}{\sin(\pi l)} dl.
\end{aligned}
\tag{23}
$$

Now we must take the large–χ (large–s) limit of this expression, with the hyperbolic angle χ expressed in terms of s by the relation (5), i.e., $\cosh\chi = \frac{s}{2m^2} - 1$. The asymptotic form of $P_\nu(z)$ when $z \to \infty$ is known to be a linear combination of z^ν and of $z^{-\nu-1}$. When $\mathrm{Re}(\nu) > -1/2$, the last term can be neglected and thus, in the limit $s \to \infty$, with a fixed t ($|t| \ll s$), we obtain, from the sum in Eq. (23) (see Ref. [18] for more details):

$$
\widetilde{\mathcal{C}}_M^{(hh)} \left(\chi \underset{s\to\infty}{\sim} \log\left(\frac{s}{m^2}\right), t \right) \sim \sum_{\mathrm{Re}(\sigma_n^+) > -\frac{1}{2}} \beta_n^+(t) s^{\sigma_n^+(t)}. \tag{24}
$$

The integral in Eq. (23), usually called the *background term*, vanishes at least as $1/\sqrt{s}$ and therefore can be neglected. From eqs. (12) and (24) we can extract the elastic scattering amplitude:

$$
\mathcal{M}_{(hh)}(s,t) \underset{s\to\infty}{\sim} -2i \sum_{\mathrm{Re}(\sigma_n^+) > -\frac{1}{2}} \beta_n^+(t) s^{1+\sigma_n^+(t)}. \tag{25}
$$

This equation gives the explicit s–dependence of the scattering amplitude at very high energy ($s \to \infty$) and small transferred momentum ($|t| \ll s$). As we can see, this amplitude comes out to be a sum of powers of s. This sort of behaviour for the scattering amplitude is known in the literature as a *Regge behaviour* and $1 + \sigma_n^+(t) \equiv \alpha_n^+(t)$ is the so–called *Regge trajectory*. In the original derivation (see, e.g., Ref. [4] and references therein) the asymptotic behaviour (25) is recovered by analytically continuing the t–channel scattering amplitude to very large imaginary values of the angle between the trajectories of the two exiting particles in the t–channel scattering process. Instead, in our derivation (see Ref. [18]), we have used the Euclidean–to–Minkowskian analytic continuation (13) and we have analytically continued the Euclidean loop–loop correlator to very large (negative) imaginary values of the angle θ between the two Euclidean Wilson loops.

Denoting with $\sigma_P(t)$ the pole with the largest real part (at that given t) and with $\beta_P(t)$ the corresponding coefficient $\beta_n^+(t)$ in Eq. (24), we thus find that:

$$
\widetilde{\mathcal{C}}_M^{(hh)} \left(\chi \underset{s\to\infty}{\sim} \log\left(\frac{s}{m^2}\right), t \right) \sim \beta_P(t) s^{\sigma_P(t)} \Longrightarrow \mathcal{M}_{(hh)}(s,t) \underset{s\to\infty}{\sim} -2i\, \beta_P(t)\, s^{\alpha_P(t)}, \tag{26}
$$

where $\alpha_P(t) \equiv 1 + \sigma_P(t)$ is the *pomeron trajectory*. Therefore, by virtue of the optical theorem:

$$
\sigma_{\mathrm{tot}}^{(hh)}(s) \underset{s\to\infty}{\sim} \frac{1}{s} \mathrm{Im}\mathcal{M}_{(hh)}(s, t=0) \sim \sigma_0^{(hh)} \left(\frac{s}{s_0}\right)^{\epsilon_P}, \quad \text{with } \epsilon_P = \mathrm{Re}[\alpha_P(0)] - 1. \tag{27}
$$

We want to stress two important issues which clarify under which conditions we have been able to derive this *pomeron*–like behaviour for the elastic amplitudes and the total cross sections.

i) We have ignored a possible energy dependence of hadron wave functions and we have thus ascribed the high–energy behaviour of the Minkowskian hadronic correlation function exclusively to the *fundamental* loop–loop correlation function (9). With this hypothesis, the coefficients A_l^+ in the partial–wave expansion (15) and, as a consequence, the coefficients β_n^+ and σ_n^+ in the Regge expansion (24) do not depend on s, but they only depend on the variable t.

ii) However, this is not enough to guarantee the experimentally–observed *universality* (i.e., independence on the specific type of hadrons involved in the reaction) of the *pomeron* trajectory $\alpha_P(t)$ in Eq. (26) and, therefore, of the *pomeron* intercept $1+\epsilon_P$ in Eq. (27). In fact, the partial–wave expansion (15) of the hadronic correlation function can also be considered as a result of a partial–wave expansion of the loop–loop Euclidean correlation function (10), i.e.,

$$\widetilde{\mathcal{C}}_E(\theta,t;1,2) = \sum_{l=0}^{\infty}(2l+1)\mathcal{A}_l(t;1,2)P_l(\cos\theta), \tag{28}$$

which is then averaged with two proper squared hadron wave functions, in the same sense as in Eq. (11), so giving the Euclidean hadronic correlation function (15). If we now repeat for the partial–wave expansion (28) the same manipulations that have led us from Eq. (15) to Eq. (24), we arrive at the following Regge expansion for the (even–signatured) loop–loop Minkowskian correlator:

$$\widetilde{\mathcal{C}}_M^+\left(\chi \underset{s\to\infty}{\sim} \log\left(\frac{s}{m^2}\right), t;1,2\right) \sim \sum_{\mathrm{Re}(a_n^+)>-\frac{1}{2}} b_n^+(t;1,2) s^{a_n^+(t;1,2)}, \tag{29}$$

where $a_n^+(t;1,2)$ is a pole of $\mathcal{A}_l^+(t;1,2)$ in the complex l–plane. After inserting the expansion (29) into the expression for the Minkowskian hadronic correlation function:

$$\begin{aligned}\widetilde{\mathcal{C}}_M^{(hh)}(\chi,t) &= \int d^2\vec{R}_{1\perp}\int_0^1 df_1\,|\psi_1(\vec{R}_{1\perp},f_1)|^2\int d^2\vec{R}_{2\perp}\int_0^1 df_2\,|\psi_2(\vec{R}_{2\perp},f_2)|^2 \\ &\times\ \widetilde{\mathcal{C}}_M^+(\chi,t;1,2), \end{aligned}\tag{30}$$

one in general finds a high–energy behaviour which hardly fits with that reported in Eq. (26) with a universal *pomeron* trajectory $\alpha_P(t)$, *unless* one assumes that, for each given loop–loop correlation function with transverse separations $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ and longitudinal–momentum fractions f_1 and f_2, (at least) the location of the pole $a_n^+(t;1,2)$ with the largest real part does not depend on $\vec{R}_{1\perp}, f_1$ and $\vec{R}_{2\perp}, f_2$, but only depends on t. If we denote this common pole with $\sigma_P(t)$, we then immediately recover the high–energy behaviour (26), where the coefficient $\beta_P(t)$ in front, differently from the universal function $\alpha_P(t) = 1+\sigma_P(t)$, explicitly depends on the specific type of hadrons involved in the process.

3 Conclusions and outlook

In conclusion, we have shown that the Euclidean–to–Minkowskian analytic–continuation approach can, with the inclusion of some extra (more or less plausible) *assumptions*, easily reproduce a *pomeron*–like behaviour for the high–energy total cross sections, in apparent agreement with the present–time experimental observations. However, we should also keep in mind that

the *pomeron*–like behaviour (14) is, strictly speaking, theoretically forbidden (at least if considered as a true *asymptotic* behaviour) by the well–known Froissart–Lukaszuk–Martin (FLM) theorem [20]. In this respect, the *pomeron*–like behaviour (14) can at most be regarded as a sort of *pre–asymptotic* (but not really *asymptotic*) behaviour of the high–energy total cross sections, valid in a certain high–energy range.

Immediately the following question arises: why our approach, which was formulated so to give the really asymptotic large–s behaviour of scattering amplitudes and total cross sections, is also able to reproduce pre–asymptotic behaviours (violating the FLM bound) like the one in (14)?

The answer is clearly that the extra *assumptions*, i.e., the *models*, which one implicitly or explicitly uses in the calculation of the Euclidean correlation function $\widetilde{\mathcal{C}}_E$, play a fundamental role in this respect. Of course, every model has its own *limitations*, which reflect in the variety of answers in the literature. Unfortunately these *limitations* are often out of control, in the sense that no one knows exactly what is losing due to these approximations. This is surely a crucial point which, in our opinion, should be further investigated in the future, also with the help of direct lattice calculations of the loop–loop Euclidean correlation function.

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